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Ladder operator method for the potential $V(x) = Ax^2 + Bx^{-2}$

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Abstract : Adopting a general procedure of constructing ladder operators – energy eigenvalue raising and lowering operators, it is shown that ladder operators which are quadratic in momentum operator p, that is, which are of the form $Q = p^2 + b(x)\,p + c(x)$, can be constructed for and only for the potential of the form $V(x) = Ax^2 + Bx^{-2}$, $B > 0$. By explicitly constructing these operators and the relevant number operator, which is not $Q^\dagger Q$, the energy eigenvalues are obtained and the normalized eigenvectors are determined in both the energy representation and the position representation. The operators Q, $Q^\dagger$ and H (the Hamiltonian) form a Lie algebra and generate the SU(1,1) Lie group.

Keywords : Ladder operators, raising and lowering operators, Kratzer modified oscillator.

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1. Introduction

Ladder operators, the eigenvalue raising and lowering operators provide an elegant algebraic technique of finding the energy eigenvalues and constructing the energy eigenstates of the harmonic oscillator system in quantum mechanics [1]. It is instructive to enquire whether there are other potential systems which would admit ladder operators. In Section 2, we briefly recapitulate the basic tenets of the ladder operator method as applied to the familiar linear harmonic oscillator system, the emphasis being on finding all the *normalized* wavefunctions (the energy eigenvectors in the position representation) without having to solve the Schrödinger equation. In Section 3, we formulate a general method of constructing ladder operators for a non-relativistic quantum system in one-dimension, described by a potential $V(x)$. Here, we explicitly show that (a) ladder operators that are linear in the momentum operator, can be constructed only for the harmonic oscillator potential $V(x) = \frac{1}{2}a^2x^2$, and (b) ladder operators which are quadratic in the momentum operator, can be constructed only for a potential of the form $V(x) = Ax^2 + Bx^{-2}$ which we call a Kratzer modified oscillator potential** [2].

Section 4 is more or less self-contained. Here, we apply the ladder operator technique to determine the energy eigenvalues and the *normalized* eigenvectors, in both the energy representation and the position representation, for the potential $V(x)\frac{1}{2} = (m\omega^2x^2 + Kx^{-2})$.

2. The ladder operator treatment of the linear harmonic oscillator

The linear harmonic oscillator is described by the Hamiltonian

$$H = \hbar\omega\frac{1}{2}\left(p^2 + x^2\right), \tag{1}$$

where p and x are dimensionless momentum and position

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**In reference [2], Kratzer had considered a central potential of the form
$V(r) = -e^2r^{-1} + \alpha r^{-2} + \beta r^{-3}$,
which is the familiar Coulomb potential modified by the addition of higher inverse-power therms. The potential $V(x) = Ax^2 + Bx^{-2}$ is the familiar harmonic oscillator potential modified by the addition of the first Kratzer term Bx^{-2}.

variables satisfying the fundamental commutation relation

$$[x, p] = i \tag{2}$$

Defining the operators as

$$a = \frac{1}{\sqrt{2}}(p - ix), \text{ and } a^{\dagger} = \frac{1}{\sqrt{2}}(p + ix), \tag{3}$$

we have

$$H = \hbar\omega\left(a^{\dagger}a + \frac{1}{2}\right) \tag{4}$$

The laddering property of $a^{\dagger}$ and a results from the commutation relations :

$$[H,a^{\dagger}] = \hbar\omega a^{\dagger}, \text{ and } [H,a] = -\hbar\omega a. \tag{5}$$

It is easy to show that the number operator $N = a^{\dagger}a$ has eigenvalue

$n = 0, 1, 2, 3, \ldots\ldots$

and hence the energy eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \tag{6}$$

The ground state eigenvector Ψ_0 is defined by

$$a\Psi_0 = 0. \tag{7}$$

In the position representation, $p = -i\frac{d}{dx}$ and hence the ground state wave function $\Psi_0(x)$ satisfies the equation

$$(d/dx + x)\Psi_0(x) = 0, \tag{8}$$

solution of which is

$$\Psi_0(x) = A_0 \exp\left(-\frac{1}{2}x^2\right). \tag{9}$$

The normalization constant A_0 is easily determined to be $(\sqrt{\pi})^{-1/2}$. The normalized higher states are then determined as

$$\Psi_n(x) = \frac{1}{\sqrt{n!}}(a^{\dagger})^n\Psi_0(x)$$

$$= \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\left(x - \frac{d}{dx}\right)^n \exp\left(-\frac{1}{2}x^2\right)$$

$$= \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}\exp\left(-\frac{1}{2}x^2\right)H_n(x). \tag{10}$$

The last step follows from the Rodrigue's formula for the Hermite polynomial $H_n(x)$ on using the identity

$$\exp\left(\frac{1}{2}x^2\right)(D - x)^n \exp\left(-\frac{1}{2}\right.$$

$$= \exp(x^2)D^n \exp(-x^2),$$

where $D = d/dx$. (11)

3. Ladder operators and potentials admitting certain types of them

For a given Hamiltonian H, if linear ioperators $Q_{\pm}$ exist such that

$$[H,Q_{\pm}] = \pm \varepsilon Q_{\pm} \tag{12}$$

where ε is a real number having dimension of energy. If $H\Psi_E = H\Psi_E$ then

$$H(Q_{\pm}\Psi_E) = (E \pm \varepsilon)Q_{\pm}Y_E. \tag{13}$$

Eq. (12) defines the ladder operators, Q_+ being the raising operators and Q_- the lowering operator. It is obvious that Q_+ and Q_- are essentially the hermitian conjugate of each other.

To explore the forms of the potentials for which we can explicitly construct the types of the ladder operators considered below, we take the non-relativistic one dimensional Hamiltonian, choosing a unit system in which $\hbar = 1$ and $m = 1$,

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$$

$$= -\frac{1}{2}D^2 + V(x). \tag{14}$$

3.1. Ladder operators which are linear in momentum :

The most general form of the raising operator Q_+ which is linear in momentum p, is

$$Q_+ = D + a(x) \tag{15}$$

apart from an inessential overall multiplicative constant. The defining equation

$$[H,Q_+] = \alpha Q_+ \tag{16}$$

gives

$$a'(x)D + \frac{1}{2}a''(x) + V'(x) = -\alpha(D + a(x)), \tag{17}$$

from which we get the following set of equations :

$$a'(x) = -\alpha$$

and $a''(x) + 2V'(x) = -\alpha a(x)$. (18)

These equations are easily solved to give

$$a(x) = -\alpha x$$

and $V(x) = \frac{1}{2}\alpha^2 x^2$, (19)

wherein we have set the inessential constants of integration equal to zero. We thus reach the conclusion that "ladder operator which is linear in momentum exists, only for the harmonic oscillator potential".

3.2. Ladder operators which are quadratic in momentum :

The most general form of a raising operator which is quadratic in the momentum p, is

$$Q_+ = D^2 + a_1(x)D + a_0(x). \quad (20)$$

Taking the defining commutation relation as

$$[H, Q_+] = 2\alpha Q_+, \quad (21)$$

we get the following set of coupled differential equations :

$$a_1'(x) = -2\alpha,$$

$$\frac{1}{2}a_1''(x) + a_0'(x) + 2V'(x) = -2\alpha a_1(x), \quad (22)$$

$$\frac{1}{2}a_0''(x) + a_1'(x) + V'(x) + V''(x) = -2\alpha a_0(x).$$

Solution of these equations determines $a_0(x)$, $a_1(x)$ and $V(x)$ as

$$a_1(x) = -2\alpha x, \quad (23a)$$

$$a_0(x) = \alpha^2 x - Bx^{-2} - \alpha, \quad (23b)$$

and $$V(x) = \frac{1}{2}\left(\alpha^2 x^2 + Bx^{-2}\right), \quad (23c)$$

where B is a real constant. We have set other inessential constants of integration equal to zero, without loss of generality. We thus find that the defining commutation relation, eq. (21), not only determines the coefficient functions $a_0(x)$ and $a_1(x)$ but also determines the potential $V(x)$ to be that of a linear harmonic oscillator modified by the addition of a Kratzer term Bx^{-2}. In other words, "ladder operators which are quadratic in momentum can be constructed only for a potential of the form $V(x) = Ax^2 + Bx^{-2}$".

4. Basic Quantum Mechanics of a particle in a potential of the form $V(x) = Ax^2 + Bx^{-2}$ through ladder operators

4.1. The Hamiltonian :

We consider a particle of mass m in a potential

$$V(x) = \frac{1}{2}\left(m\omega^2 x^2 + Kx^{-2}\right), \; K > 0 \quad (24)$$

which is a harmonic oscillator subjected to a repulsive force Kx^{-3}.

Measuring momentum in units of $\sqrt{m\hbar\omega}$ and position coordinate in units of $(\hbar/m\omega)^{1/2}$, the Hamiltonian can be written as

$$H = \hbar\omega\frac{1}{2}\left(p^2 + x^2 + \beta(\beta-1)x^{-2}\right), \quad (25)$$

where β is a dimensionless real parameter and now onwards, p and x are dimensionless momentum and position variables satisfying the commutation relation

$$[x, p] = i. \quad (26)$$

Construction of ladder operators :

Defining the operators

$$A = 1/\sqrt{2}\,(p - ix + i\beta x^{-1}) \quad (27a)$$

and $$B = 1/\sqrt{2}\,(p - ix - i\beta x^{-1}), \quad (27b)$$

we get $$H = \hbar\omega A^\dagger A + E_0, \quad (28a)$$

and also $$H = \hbar\omega BB^\dagger A - E_0, \quad (28b)$$

with $$E_0 = \left(\beta + \frac{1}{2}\right)\hbar\omega. \quad (29)$$

We now define the operators

$$Q = BA \text{ and } Q^\dagger = A^\dagger B^\dagger \quad (30)$$

and find that they satisfy the commutation relations

$$[H, Q^\dagger] = 2\hbar\omega Q^\dagger \quad (31a)$$

and $$[H, Q^\dagger] = -2\hbar\omega Q. \quad (31b)$$

Thus, the operators $Q^\dagger$ and Q are the ladder operators, respectively raising and lowering the energy eigenvalues by $2\hbar\omega$.

4.2. The energy eigenvalues and the number operator :

If Ψ_E is an eigenvector of H belonging to the eigenvalue E, then from eq. (28a), we get

$$\hbar\omega\left\langle\Psi_E\middle|A^\dagger A\middle|\Psi_E\right\rangle = \left\langle\Psi_E\middle|H - E\middle|\Psi_E\right\rangle$$

that is,

$$\hbar\omega\langle A\Psi_E \mid A\Psi_E\rangle = (E - E_0)\langle\Psi_E \mid \Psi_E\rangle.$$

Since the norms are non-negative, it follows that

$$E \geq E_0. \quad (32)$$

That is, E_0 is the ground state energy. And when $E = E_0$ (for the ground state), we have

$$\left\|A\Psi_{E_0}\right\| = 0.$$

That is, $$A\Psi_{E_0} = 0. \quad (33)$$

Eq. (33) shall be taken as the definition of the ground state, and it follows that the lowering operator Q also annihilates the ground state. Successive application of $Q^\dagger$ on the ground state would then generate the higher energy eigenstates :

$$H(Q^\dagger)^n\Psi_{E_0} = (E_0 + 2n\hbar\omega)(Q^\dagger)^n\Psi_{E_0}. \quad (34)$$

Thus, the energy eigenvalues are obtained as

$$E_n = E_0 + 2n\hbar\omega, \; n = 0, 1, 2, \ldots\ldots \tag{35}$$

Defining $N = \frac{1}{2}A^\dagger A$, (36)

we have $H = E_0 + 2\hbar\omega N$ (37)

and $[N, Q^\dagger] = Q^\dagger$ and $[N, Q] = -Q$.

These commutation relations mean that N is the number operator.

In fact,

$N\Psi_{E_n} = n\Psi_{E_n}$ where $H\Psi_{E_n} = E_n\Psi_{E_n}$.

We thus identify N as the number operator even though neither A is the lowering operator nor is its conjugate $A^\dagger$ the raising operator. Further $Q^\dagger Q$ is not the number operator.

4.3. The normalized eigenstates :

It is easy to show that

$$QQ^\dagger = 4(N+1)\left(N+\beta+\frac{1}{2}\right) \tag{38}$$

and $Q^\dagger Q = 4N\left(N+\beta-\frac{1}{2}\right)$, (39)

so that

$$[Q, Q^\dagger] = 4H/\hbar\omega. \tag{40}$$

Denoting the normalized energy eigenvectors by $|n\rangle$, $n = 0, 1, 2.....$,

we have

$$H|n\rangle = E_n|n\rangle \; ; \; \langle m|n\rangle = \delta_{mn},$$

$$N|n\rangle = n|n\rangle,$$

and $Q^\dagger|n\rangle = c_n|n+1\rangle$,

$$Q|n\rangle = d_n|n-1\rangle.$$

Then, $QQ^\dagger|n\rangle = 4(n+1)|n+\beta+$

and $|d_n|^2 = \langle n|Q^\dagger Q|n\rangle = 4\left(n+\beta-\frac{1}{2}\right)$.

Taking the real positive roots for c_n and d_n, we have

$$|n+1\rangle = \left[4(n+1)\left(n+\beta+\frac{1}{2}\right)\right]^{-1/2} Q^\dagger|n\rangle \tag{41}$$

and

$$|n-1\rangle = \left[4n\left(n+\beta-\frac{1}{2}\right)\right]^{-1/2} Q|n\rangle. \tag{42}$$

Eq. (41) is used to construct the normalized eigenvectors

$$|n\rangle = \frac{1}{2^n\sqrt{n!}}\left|\frac{\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma\left(n+\beta+\frac{1}{2}\right)}\right|^{1/2}(Q^\dagger)^n|0\rangle \tag{43}$$

4.4. The wavefunctions :

The singularity of the potential at $x = 0$, divides the space into two disjoint regions, $x > 0$ and $x < 0$. Only one of the two disjoint regions is accessible to the particle. We choose the region $x > 0$. The ground state wavefunction satisfies eq. (33), which in the position representation is

$$(d/dx + \lambda - \beta/x)\Psi_0(x) = 0.$$

Its solution satisfying the normalization condition $\int_0^\infty |\Psi_0|^2 dx = 1$ is

$$\Psi_0(x) = \left[\frac{1}{2}\Gamma\left(\beta+\frac{1}{2}\right)\right]^{-1/2} x^\beta \exp\left(-\frac{1}{2}x^2\right). \tag{44}$$

In the position representation

$$Q^\dagger = \frac{H}{\hbar\omega} + x\frac{d}{dx} - x^2 + \frac{1}{2} \tag{45}$$

and hence eq. (41) becomes

$$\Psi_{n+1}(x) = \frac{1}{2\sqrt{(n+1)\left(n+\frac{1}{2}+\beta\right)}}\left(2n+\beta+1-x^2+x\frac{d}{dx}\right)\Psi_n(x). \tag{46}$$

Knowing the normalized groundstate wavefunction $\Psi_0(x)$ eq. (44), we can use eq. (46) to successively generate the normalized wavefunctions of the higher states. We can thus, show that the normalized energy eigenfunctions are

$$\Psi_n(x) = A_n x^\beta \exp\left(-\frac{1}{2}x^2\right) {}_1F_1\left(-n, \beta+\frac{1}{2}; x^2\right), \tag{47}$$

where ${}_1F_1(a, c; z)$ is the standard confluent or degenerate hypergeometric function [3], and the normalization constant

$$A_n = \frac{1}{\Gamma\left(\beta+\frac{1}{2}\right)}\sqrt{\frac{2\Gamma\left(\beta+\frac{1}{2}+n\right)}{n!}}. \tag{48}$$

That the function $\Psi_n(x)$ given in eq. (47) with the constant A_n given by eq. (48) satisfies the recursion relation (46), can be easily established with the help of the following recurrence relation of the confluent hypergeometric functions

$$[n + c - z + z(d/dz)]_1F_1(-n, c; z) = (n + c)_1F_1(-n - 1, c; z), \quad (49)$$

which can be easily proved from the series definition of ${}_1F_1(a, c; z)$.

5. Discussion

(i) Existence of ladder operators implies that the Hamiltonian has equispaced eigenvalue spectrum. Such Hamiltonians are quite restrictive in the sense that the only known potential exhibiting equispaced level structure is that of the harmonic oscillator with or without the Kratzer modification term Bx^{-2}.

Supersymmetric partner Hamiltonians [4] have identical eigenvalue spectrum except for the ground state. The supersymmetric partner of the Hamiltonian

$H_1 = \hbar\omega A^\dagger A + E_{01}$ (*cf.* eq.(28a)) (here, $E_{01} = \hbar\omega\left(\beta + \frac{1}{2}\right)$ is $H_2 = \hbar\omega AA^\dagger + E_{01}$.

In terms of the potentials, the supersymmetric partner of the potential

$$V_1(x) = \hbar\omega\frac{1}{2}\left(x^2 + \beta(\beta-1)x^{-2}\right)$$

is the potential

$$V_2(x) = \hbar\omega\frac{1}{2}\left(x^2 + \beta(\beta+1)x^{-2}\right),$$

which is of the same form as $V_1(x)$ and obviously has the same spectrum as $V_1(x)$ with ground state energy $E_{02} = E_{01} + 2\hbar\omega$.

(ii) The appearance of the x^{-1} terms in the expressions for the auxiliary operators A and B, eqs. (27a) and (27b) may on first sight cast a doubt on the boundedness of the ladder operators Q and $Q^\dagger$. Such a doubt is removed by observing that

$$Q = H/\hbar\omega - ixp - x^2 - \frac{1}{2}$$

and

$$Q^\dagger = H/\hbar\omega - ixp - x^2 + \frac{1}{2},$$

which are well behaved bounded operators in the space spanned by the eigenvectors of the Hamiltonian H. In fact, eq.(38) gives

$$\left\|Q^\dagger\Psi_n\right\| = \sqrt{4(n+1)\left(n+\beta+\frac{1}{2}\right)}$$

which directly shows the boundedness of $Q^\dagger$. Similarly boundedness of Q follows directly from eq. (39). The boundedness of the auxilliary operators A and B follows from eqs. (28a) and (28b).

(iii) The reduced radial Schrödinger equation for a central potential $V(r) = Ar^2 + Br^{-2}$ is identical in form, to the one dimensional Schrödinger eigenvalue equation for the potential $V(x) = \frac{1}{2}(m\omega^2x^2 + Kx^{-2})$. Therefore, the energy eigenvalues and *normalized* wavefunctions obtained in Section 4, can be readily adapted for this central potential without having to solve the Schrödinger equation. An added advantage of this method over the method of direct solution of the Schrödinger equations [5,6] is that the normalization constants of the wavefunctions are automatically determined once we normalise the groundstate wavefunction.

(iv) Defining the dimensionless operators

$\Gamma_1 = \frac{1}{4}(Q + Q^\dagger)$, $\Gamma_2 = \frac{1}{4}(Q - Q^\dagger)$ and $\Gamma_3 = H/2\hbar\omega$, it is easy to verify (using eqs. (31a), (31b) and (40)) that

$$[\Gamma_1, \Gamma_2] = -i\Gamma_3, \quad [\Gamma_2, \Gamma_3] = i\Gamma_1, \quad [\Gamma_3, \Gamma_1] = i\Gamma_2,$$

which is the Lie algebra of the generators of SU(1,1) and SO(2,1) groups [7,8] . In other words, the operators Γ_1 , Γ_2 and Γ_3 form a second order realization [9] of the SU(1,1) Lie algebra. Thus, Q and $Q^\dagger$ are essentially the non-compact generators and H the compact generator of SU(1,1) group.

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